Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; "THE THIRD FLOOR", Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most solvers used the facts that $f(1) = 0$ and $f'(1) = -1$ to deduce that $f(1+\varepsilon) < 0$ for sufficiently small $\varepsilon > 0$.

Lau showed that f has only one root exceeding one, and Woo established the stronger result that in fact, $r \in \left(1, 1 + \frac{1}{n(n-1)}\right)$.

3182. Replacement. [2007 : 40, 43] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be any positive real numbers, and let p be a real number such that $0 < p < 1$.

(a) Prove that

$$\frac{a}{(b+c)^p} + \frac{b}{(c+a)^p} + \frac{c}{(a+b)^p} \ \geq\ \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right) .$$

(b) Prove that, if $p = 1/3$, then

$$\frac{a}{(a+b)^p} + \frac{b}{(b+c)^p} + \frac{c}{(c+a)^p} \ \geq\ \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right) .$$

(c)★ Prove or disprove

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ \geq\ \frac{1}{\sqrt{2}}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) .$$

Solution to part (a) *by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Since the proposed inequality is homogeneous, we may suppose that $a + b + c = 1$. This yields the equivalent inequality

$$\frac{a}{(1-a)^p} + \frac{b}{(1-b)^p} + \frac{c}{(1-c)^p} \ \geq\ \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right) .$$

Witout loss of generality, we may assume that $a \geq b \geq c$. It then follows that

$$\frac{1}{1-a} \ \geq\ \frac{1}{1-b} \ \geq\ \frac{1}{1-c} .$$

Using Chebyshev's Inequality and the AM–GM Inequality, we have

$$\begin{aligned}
&\frac{a}{(1-a)^p}+\frac{b}{(1-b)^p}+\frac{c}{(1-c)^p}\\
&\quad\geq\ \frac{a+b+c}{3}\left[\frac{1}{(1-a)^p}+\frac{1}{(1-b)^p}+\frac{1}{(1-c)^p}\right]\\
&\quad\geq\ \frac{1}{\big((1-a)(1-b)(1-c)\big)^{p/3}}\ \geq\ \frac{1}{\left[\frac{1-a+1-b+1-c}{3}\right]^p}\ =\ \frac{3^p}{2^p}.
\end{aligned}$$

Thus, we need only prove that $a^{1-p}+b^{1-p}+c^{1-p}\leq 3^p$.

Let $f(x)=x^{1-p}$. Since $f''(x)=-p(1-p)x^{-p-1}$ and $0<p<1$, we see that $f''(x)<0$ for $0<x<1$. Hence, f is a concave function on $(0,1)$. Using Jensen's Inequality, we get

$$a^{1-p}+b^{1-p}+c^{1-p}\ \leq\ 3f\left(\tfrac{1}{3}(a+b+c)\right)\ =\ 3f\left(\tfrac{1}{3}\right)\ =\ 3^p\,.$$

[*Ed.:* Equality holds if and only if $a=b=c$.]

Solution to part (b) *by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

We will first prove that, for all $x>0$, the following inequality holds

$$\frac{4x^3\sqrt[3]{2}}{\sqrt[3]{x^3+1}}\ \geq\ 5x^2-1\,. \tag{1}$$

This inequality is trivial if $x\leq 1/\sqrt{5}$. For $x>1/\sqrt{5}$, define

$$f(x)\ =\ \frac{128x^9}{(x^3+1)(5x^2-1)^3}\,.$$

Proving inequality (1) is equivalent to showing that $f(x)\geq 1$. We compute

$$f'(x)\ =\ \frac{384x^8(1-x)(2x^2-3x-3)}{(x^3+1)^2(5x^2-1)^4}\,,$$

from which we see that $f'(x)>0$ for $1<x<\frac{1}{4}(3+\sqrt{33})$ and $f'(x)<0$ for $1/\sqrt{5}<x<1$ and for $x>\frac{1}{4}(3+\sqrt{33})$. Thus,

$$f(x)\ \geq\ \min\left\{f(1),\ \lim_{x\to\infty}f(x)\right\}\ =\ \min\left\{1\,,\frac{128}{125}\right\}\ =\ 1$$

for all $x>1/\sqrt{5}$. Therefore, inequality (1) holds for all $x>0$.

Replacing x in (1) by $\sqrt[3]{a/b}$, $\sqrt[3]{b/c}$, and $\sqrt[3]{c/a}$ in turn yields:

$$\frac{a\sqrt[3]{2}}{\sqrt[3]{a+b}}\geq\frac{5a^{\frac{2}{3}}-b^{\frac{2}{3}}}{4}\,,\quad \frac{b\sqrt[3]{2}}{\sqrt[3]{b+c}}\geq\frac{5b^{\frac{2}{3}}-c^{\frac{2}{3}}}{4}\,,\quad \frac{c\sqrt[3]{2}}{\sqrt[3]{c+a}}\geq\frac{5c^{\frac{2}{3}}-a^{\frac{2}{3}}}{4}\,.$$

Adding these inequalities produces the desired result. Equality holds if and only if $a=b=c$.

Solution to part (c) *by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

If we set $a = x^2$, $b = y^2$, and $c = z^2$, the proposed inequality becomes

$$\sum_{\text{cyclic}} \frac{x^2}{\sqrt{x^2+y^2}} \;\geq\; \frac{x+y+z}{\sqrt{2}}\,.$$

By squaring both sides, we can rewrite the inequality as

$$\sum_{\text{cyclic}} \frac{x^4}{x^2+y^2} + 2\sum_{\text{cyclic}} \frac{x^2y^2}{\sqrt{(x^2+y^2)(y^2+z^2)}} \;\geq\; \frac{(x+y+z)^2}{2}\,.$$

By the Rearrangement Inequality,

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{x^2y^2}{\sqrt{(x^2+y^2)(y^2+z^2)}} &= \sum_{\text{cyclic}} \frac{x^2y^2}{\sqrt{x^2+y^2}} \cdot \frac{1}{\sqrt{y^2+z^2}} \\
&\geq \sum_{\text{cyclic}} \frac{x^2y^2}{\sqrt{x^2+y^2}} \cdot \frac{1}{\sqrt{x^2+y^2}} \\
&= \sum_{\text{cyclic}} \frac{x^2y^2}{x^2+y^2}\,.
\end{aligned}$$

Thus, it suffices to prove that

$$\sum_{\text{cyclic}} \frac{x^4}{x^2+y^2} + 2\sum_{\text{cyclic}} \frac{x^2y^2}{x^2+y^2} \;\geq\; \frac{(x+y+z)^2}{2}\,. \tag{2}$$

Moreover, we observe that

$$\sum_{\text{cyclic}} \frac{x^4-y^4}{x^2+y^2} \;=\; \sum_{\text{cyclic}} (x^2-y^2) \;=\; 0\,.$$

Hence,

$$\sum_{\text{cyclic}} \frac{x^4}{x^2+y^2} \;=\; \frac{1}{2}\sum_{\text{cyclic}} \frac{x^4+y^4}{x^2+y^2}\,.$$

Therefore, inequality (2) is successively equivalent to

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{x^4+y^4}{x^2+y^2} + \sum_{\text{cyclic}} \frac{4x^2y^2}{x^2+y^2} &\geq (x+y+z)^2\,, \\
\sum_{\text{cyclic}} \frac{x^2+y^2}{2} + \sum_{\text{cyclic}} \frac{2x^2y^2}{x^2+y^2} &\geq 2\sum_{\text{cyclic}} xy\,, \\
\sum_{\text{cyclic}} \frac{(x-y)^4}{2(x^2+y^2)} &\geq 0\,,
\end{aligned}$$

which is clearly true. Therefore, the inequality holds. Equality holds if and only if $a = b = c$.

All three parts were also solved by SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and VO QUOC BA CAN. Part (a) *was solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. Part* (c) *was also solved by CAO MINH QUANG. The proposer solved parts* (a) *and* (b).

There were several solvers who had submitted correct solutions to the original 3182, which was the same as 3096 [2005 : 544, 547; 2006 ; 531]. Most of them were already listed there as having solved 3096. However, CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and D. KIPP JOHNSON, Beaverton, OR, USA should be added to that list.

3183. [2006 : 463, 464] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a triangle with inradius r and circumradius R. If s is the semiperimeter of the triangle, prove that

$$\sqrt{3}\,s \;\le\; r + 4R\,.$$

Remark by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and D. Kipp Johnson, Beaverton, OR, USA.

This is a very old problem. Its origin is referred back to the year 1872 in [1, item 5.5].

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 *solutions*)*; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

3184. [2006 : 463] *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For any real number x, let (x) denote the *fractional part* of x; that is, $(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Given $n \in \mathbb{Z}$, find all solutions of the equation

$$(x^2) - n(x) \;=\; 0\,.$$

Solution by Michel Bataille, Rouen, France.

[*Ed:* We will use $\{x\}$ instead of (x) to denote the fractional part of x.]

Let S_n denote the set of all real solutions to the equation

$$\{x^2\} - n\{x\} \;=\; 0\,. \qquad (1)$$

Note that $\mathbb{Z} \subseteq S_n$, since $\{k^2\} = \{k\} = 0$ for all $k \in \mathbb{Z}$.